Appendix A
Errors

Measurement errors

In Section 1.3 the idea of accuracy and errors associated with instrumentation were discussed. Here we discus other sources of error and how all such errors affect the accuracy of measurements.

When a physical quantity is measured, the value obtained should not be expected to be exactly what the quantity actually is. With every measured quantity there is associated some error. These errors can arise from:

1 *Instrumentation errors*
Such errors are an inherent feature of an instrument and can arise in the manufacture of the instrument from such causes as tolerances on the dimensions of mechanical components and the values of electrical components used in the construction of the instrument. In addition there can be errors due to other factors such as the accuracy with which the instrument has been calibrated or readings being taken under different conditions to which the instrument was calibrated, e.g. at a different temperature, non-linearity and hysteresis.

2 *Reading errors*
Such errors arise due to the limited accuracy with which scales can be read. When the pointer of an analogue instrument falls between two scale markings (Figure ApA.1) there is some uncertainty as to what the reading should be quoted as. Thus analogue instrument readings should not be quoted as precise numbers but some indication given of the uncertainty with which a reading is made and hence the extent to which the reading could be in error. The worse the reading error could be is that the value indicated by a pointer is somewhere between two successive markings on the scale. In such circumstances the reading error can be quoted as ± half the scale interval. Thus with a rule with scale readings every millimetre, the reading error might be quoted as ± 0.5 mm. Thus when using such a rule to make a measurement, if the nearest mark on the rule is, say, 65 mm, then the reading would be quoted as 65 ± 0.5 mm.

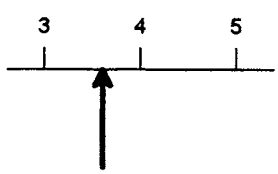

Figure ApA.1 *Reading error*

With digital displays there is no uncertainty regarding the value displayed but there is still an error associated with a reading. This is because the reading of a digital instrument goes in jumps, a whole digit at a time. It is not possible to tell where the value is between two successive digits. Thus the reading error can be quoted as ± the

smallest digit. For example a digital display giving a reading of, say, 6789 would have an error of ±1 and be quoted as 6789 ± 1.

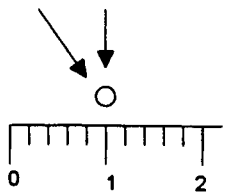

Figure ApA.2 *The effect of parallax: viewed directly from above the reading is 1.0 while viewed at an angle to the vertical the reading can be 1.2*

3. *Human errors*
   These include errors such as misreading of the position of a pointer on a scale or, where there are multiple scales, the misreading of scales. If the scale and the pointer of an instrument are not in the same plane (Figure ApA.2), then the reading obtained depends on the angle at which the pointer is viewed against the scale; such errors are termed *parallax errors*. To reduce the chance of such errors occurring, some instruments incorporate a mirror alongside the scale so that the scale is read when the pointer and its image are superimposed, thus ensuring that the pointer is being viewed at right angles to the scale. Other potential sources of human errors are the varying reaction times of individuals in timing events or applying varying pressures when using a micrometer screw gauge.

4. *Insertion errors*
   In some measurements, the insertion of the instrument into the position to measure a quantity can affect the value obtained (see Section 1.1.3). For example, inserting an ammeter into a circuit to measure the current can change the value of the current in the circuit due to the ammeter's own resistance. Similarly, putting a cold thermometer into a hot liquid to measure its temperature can cool the liquid and so change the temperature being measured.

Random errors

All errors, whatever their source, can be described as being either random or systematic. *Random errors* are ones which can vary in a random manner between successive readings of the same quantity. These may be due to personal errors by the person making the measurements or perhaps due to random electronic fluctuations (termed noise) in the instruments or circuits used, or perhaps varying frictional effects. *Systematic errors* are errors which do not vary from one reading to another. These may be due to some defect in the instrument such as a wrongly set zero so that it always gives a high or low reading, or perhaps incorrect calibration, or perhaps an instrument is temperature dependent and the measurement is made under conditions which differ from those for which it was calibrated, or there is an insertion error.

> Random errors can be minimised by taking a number of readings and obtaining a mean value, systematic errors require the use of a different instrument or measurement technique to establish them.

Mean values

Random errors give a reading that is sometimes too high, sometimes too low. The mean (average) value of a set of results will thus give a more

accurate value than just a single value. The *mean* or *average* of a set of readings is given by:

$$\bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n}$$

where x_1 is the first reading, x_2 the second reading, ... x_n the nth reading. The more readings we take the more likely it will be that we can cancel out the random variations that occur between readings.

> The *true value* of a measurement might be regarded as the value given by the mean of a very large number of readings of the particular variable concerned.

Standard deviation

Consider the two following sets of time readings:

20.1, 20.0, 20.2, 20.1, 20.1 and 19.5, 20.5, 19.7, 20.6, 20.2 s

Both sets of readings have the same average of 20.1 s, but the second set of readings is more spread out than the first and thus shows more random fluctuations. The *deviation* of any one reading from the mean is the difference between its value and the mean value. Thus the first set of readings has deviations of 0.0, –0.1, +0.1, 0.0 and 0.0; the second set has deviations of –0.6, +0.4, –0.4, +0.5 and + 0.1. The second set of readings has greater deviations than the first set and thus if we had only considered one reading of the less spread out set of readings it would have had a greater chance of being closer to the mean value than any one reading in the more spread out set.

> The spread of a set of readings is taken as a measure of the certainty we can attach to any one reading being close to the mean value, the bigger the spread the greater the uncertainty. The spread of the readings is specified by a quantity termed the *standard deviation*.

The standard deviation is given by:

$$\text{standard deviation} = \sqrt{\frac{(d_1^2 + d_2^2 + \ldots d_n^2)}{n-1}}$$

where d_1 is the deviation of the first result from its average, d_2 the deviation of the second reading, ... d_n the deviation of the nth reading from the average. Note that sometimes the above equation is written with just n instead of $(n - 1)$ on the bottom line. With just n it is assumed that the deviations are all from the true value, i.e. the mean when there are very large numbers of readings. With smaller numbers of readings, the

deviations are taken from the mean value without assuming that it is necessarily the true value. To allow for this, $(n - 1)$ is used. In fact, with more than a very few readings, the results using n and $(n - 1)$ are the same, to the accuracy with which the standard deviations are usually quoted. Table ApA.1 illustrates the calculation of the standard deviation for the two sets of results.

Table ApA. 1 *Calculation of standard deviations*

1st set of readings

Reading (s)	Deviation (s)	(Deviation)^2 s^2
20.1	0.0	0.00
20.0	−0.1	0.01
20.2	+0.1	0.01
20.1	0.0	0.00
20.1	0.0	0.00

Sum of (deviation)^2 = 0.02, hence standard deviation = $\sqrt{(0.02/4)}$ = 0.07.

2nd set of readings

Reading (s)	Deviation (s)	(Deviation)^2 s^2
19.5	−0.6	0.36
20.5	+0.4	0.16
19.7	−0.4	0.16
20.6	+0.5	0.25
20.2	+0.1	0.01

Sum of (deviation)^2 = 0.94, hence standard deviation = $\sqrt{(0.94/4)}$ = 0.48.

The first set of readings has a standard deviation of 0.07 and the second set 0.48. The second set of readings has thus a much greater standard deviation than the first set, indicating the greater spread of those results. A consideration of the statistics involved shows that we can reasonably expect about 68.3% of the readings will lie within plus or minus one standard deviation of the mean, 95.45% within plus or minus two standard deviations and 99.7% within plus or minus three standard deviations.

Error of a mean

If we take a set of readings and obtain a mean, how far from the true value might we expect the mean to be? Essentially what we do is consider the mean value of our set of results to be one of the many mean values which can be obtained from the very large number of results and calculate its standard deviation from that true value mean. To avoid confusion with the standard deviation of a single result from a mean of a

set of results, we use the term *standard error* for the standard deviation of the mean from the true value.

> The extent to which we might expect the mean of a set of readings to depart from the true value is given by the *standard error of the mean.*

The standard error is given by:

$$\text{standard error} = \frac{\text{standard deviation of the set of results}}{\sqrt{n}}$$

We can reasonably expect that there is a 68.3% chance that a particular mean value lies within plus or minus one standard error of the true value, a 95.45% chance within plus or minus two standard errors and a 99.7% chance within plus or minus three standard errors. The greater the number of measurements made, the smaller will be the standard error. Note that such a reduction in error is only a reduction in the random error, there may still be an unaffected systematic error.

Thus, for the data in Table ApA.1, the first set of measurements had a standard deviation of 0.07 s and so a standard error of $0.07/\sqrt{5} = 0.03$ s. Thus, with the mean value of 20.1 s, the chance of the true value being within ±0.03 s of 20.1 is about 68%. The chance of the true value being within ±0.06 s is about 95%. Mean values are generally quoted with the 95% chance and thus would be quoted, for the first set of measurements, as 20.1 ± 0.06 s. For the second set of measurements, the standard deviation was 0.48 s and so there is a standard error of $0.48/\sqrt{5} = 0.21$ s. The mean is 20.1 s and the chance of the true value being within ±0.21 s is 68%, within ±0.42 s about 95%. The two sets of measurements are thus likely to be written as: 20.1 ± 0.06 s and 20.1 ± 0.42 s.

Combination of errors

The determination of the value of some quantity might require several measurements to be made and their values inserted into an equation. For example, in a determination of the density ρ of a solid, measurements might be made of the mass m of the body and its volume V and the density calculated from m/V. The mass and volume measurements will each have errors associated with them. How then do we determine the consequential error in the density? This type of problem is very common. The following illustrates how we can determine the worst possible error in such situations.

Errors when adding quantities

Consider the calculation of the quantity Z from two measured quantities A and B where $Z = A + B$. If the measured quantity A has an error $\pm\Delta A$ and the quantity B an error $\pm\Delta B$ then the worst possible error we could have in Z is if the quantities are at the extremes of their error bands and the two errors ΔZ are both positive or both negative. Then we have:

$$Z + \Delta Z = (A + \Delta A) + (B + \Delta B)$$

$$Z - \Delta Z = (A - \Delta A) + (B - \Delta B)$$

Subtracting one equation from the other gives the worst possible error as:

$$\Delta Z = \Delta A + \Delta B$$

> When we add two measured quantities the worst possible error in the calculated quantity is the sum of the errors in the measured quantities.

Errors when subtracting quantities

If we have the calculated quantity Z as the difference between two measured quantities, i.e. $Z = A - B$, then, in a similar way, we can show that the worst possible error is given by

$$Z + \Delta Z = (A + \Delta A) - (B + \Delta B)$$

$$Z - \Delta Z = (A - \Delta A) - (B - \Delta B)$$

and so subtracting the two equations gives the worst possible error as:

$$\Delta Z = \Delta A + \Delta B$$

> When we subtract two measured quantities the worst possible error in the calculated quantity is the sum of the errors in the measured quantities.

Errors when multiplying quantities

If we have the calculated quantity Z as the product of two measured quantities A and B, i.e. $Z = AB$, then we can calculate the worst error in Z as being when the quantities are both at the extremes of their error bands and the errors in A and B are both positive or both negative:

$$Z + \Delta Z = (A + \Delta A)(B + \Delta B) = AB + B\Delta A + A\Delta B + \Delta A \Delta B$$

The errors in A and B are small in comparison with the values of A and B so we can neglect the quantity $\Delta A \Delta B$ as being insignificant. Then:

$$\Delta Z = B\Delta A + A\Delta B$$

Dividing through by Z gives:

$$\frac{\Delta Z}{Z} = \frac{B\Delta A + A\Delta B}{Z} = \frac{B\Delta A + A\Delta B}{AB} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

Thus, when we have the product of measured quantities, the worst possible fractional error in the calculated quantity is the sum of the fractional errors in the measured quantities. If we multiply the above equation by 100 then:

> The percentage error in the product of two measurements is equal to the sum of the percentage errors in each of the measured quantities.

If we have the square of a measured quantity, then all we have is the quantity multiplied by itself and so the error in the squared quantity is just twice that in the measured quantity. If the quantity is cubed then the error is three times that in the measured quantity.

Errors when dividing quantities

If the calculated quantity is obtained by dividing one measured quantity by another, i.e. $Z = A/B$, then the worst possible error is given when we have the quantities at the extremes of their error bands and the error in A positive and the error in B negative, or vice versa. Then:

$$Z + \Delta Z = \frac{A + \Delta A}{B - \Delta B} = \frac{A\left(1 + \frac{\Delta A}{A}\right)}{B\left(1 - \frac{\Delta B}{B}\right)}$$

Using the binomial series we can write this as:

$$Z + \Delta Z = \frac{A}{B}\left(1 + \frac{\Delta A}{A}\right)\left(1 + \frac{\Delta B}{B} - \ldots\right)$$

Neglecting products of ΔA and ΔB and writing A/B as Z, gives:

$$Z + \Delta Z = Z\left(1 + \frac{\Delta A}{A} + \frac{\Delta B}{B}\right)$$

Hence:

$$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

The worst possible fractional error in the calculated quantity is the sum of the fractional errors in the measured quantities or, if expressed in percentages:

> The percentage error in the result of the division of two measurements is equal to the sum of the percentage errors in each of the measured quantities.

Example of error calculation

The above rules for determining errors when measurements are combined can be summed up:

1 When measurements are added or subtracted, the resulting worst error is the sum of the errors.

2 When measurements are multiplied or divided, the resulting worst percentage error is the sum of the percentage errors.

The following examples illustrate the use of the above to determine the worst errors.

Example

The distance between two points is determined from the difference between two length measurements. If these are 120 ± 0.5 mm and 230 ± 0.5 mm, what will be the error in the distance?

Adding the errors gives the difference as 110 ± 1.0 mm.

Example

The resistance R of a resistor is determined from measurements of the potential difference V across it and the current I through it, the resistance being given by V/I. The potential difference has been measured as 2.1 ± 0.2 V and the current measured as 0.25 ± 0.01 A. What will be the error in the resistance?

The percentage error in the voltage reading is (0.2/2.1) × 100% = 9.5% and in the current reading is (0.01/0.25) × 100% = 4.0%. Thus the percentage error in the resistance is 9.5 + 4.0 = 13.5%. Since we have V/I = 8.4 Ω and 13.5% of 8.4 is 1.1, then the resistance is 8.4 ± 1.1 Ω.

Example

The cross-sectional area A of a wire is to be determined from a measurement of the diameter d, being given by $A = \frac{1}{4}\pi d^2$. The diameter is measured as 2.5 ± 0.1 mm. What will be the error in the area?

The percentage error in d^2 will be twice the percentage error in d. Since the percentage error in d is ±4% then the percentage error in d^2, and hence A since the others are pure numbers, is ±8%. Since $\frac{1}{4}\pi d^2$ = 4.9 mm^2 and 8% of this value is 0.4 mm^2, the result can be quoted as 4.9 ± 0.4 mm^2.